DETERMINATION OF THE VELOCITY POTENTIAL AND FORCES ON A BODY OF ROTATION LEAVING THE PLANE OF A WALL

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The body of rotation is replaced by a system of sources and dipoles. The potential of the resulting velocity and the forces on the emergent body are found approximately.

We shall discuss the vertical emergence of a body of rotation from a solid wall. We assume the liquid is inviscid and incompressible, and that the resulting flow is potential. The wall is assumed to be an infinite plane moving with finite velocity V_∞ (Fig. 1). Since the problem of determining the potential of the motion normal to the wall and that of the motion in conjunction with the wall can be treated independently, we have two problems:

1) the axisymmetric problem of the motion of a body of rotation normal to the wall with velocity V_0 is solved by introducing a distribution of sources along the longitudinal x-axis;

2) the problem of the motion of the body in conjunction with the wall with velocity V_∞ – motion without axial symmetry, but still potential motion – is solved by introducing dipoles with moments parallel to the velocity of motion V_∞.

As we know from hydromechanics [1], the source strengths can be found from the condition that the outline of the body coincides with the zero stream line. Let q(x, t) be the required source strength. On the flow due to the distribution of sources along the x-axis we superpose the incident steam and then we have the total potential:

$$\varphi = V_\infty x - \frac{1}{4\pi}\int_0^{\alpha(t)} \frac{q(t,\ \xi)\, d\xi}{\sqrt{(x-\xi)^2+\rho^2}}\ , \tag{1}$$

where $[0, \alpha]$ is the segment in which the singularities are distributed ($\alpha(t)$ is a function of the time), and $\rho = \sqrt{y^2 + z^2}$.

We may demand that at each moment of time a generator of the body of rotation be the zero stream line. Thus, the first problem is solved by the integral equation

$$-\frac{1}{2} V_0 y^2 + \frac{1}{4\pi}\int_0^{\alpha} \frac{(x-\xi)\, q(\xi)\, d\xi}{\sqrt{y^2+(x-\xi)^2}} - \frac{1}{4\pi}\int_0^{\alpha} q(\xi)\, d\xi = 0. \tag{2}$$

Since, as the body emerges from the wall there is a variable volume of the body in the fluid, we have to assume that the total source strength is nonzero. By [2], the total source strength is equal to the rate of increase of the volume of the body of rotation, i.e.,

$$\int_0^{\alpha} q(x)\, dx = \frac{dU}{dt}\ .$$

To take account of the effect of the wall we shall discuss not only the fundamental distribution of sources, but also the distribution of sources symmetrically reflected in the y-axis. For the outline of the

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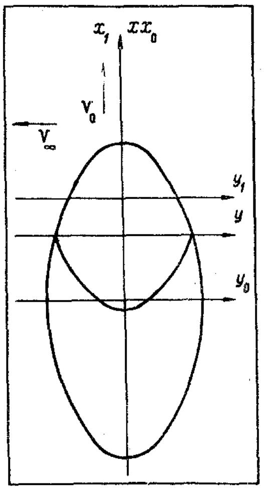

Fig. 1. The coordinate systems used in the paper.

body we take the equation of the part which has emerged together with that of the symmetrical part with respect to the y-axis.

In addition, the motion of the wall is taken into account by a distribution of dipoles with moments parallel to the y-axis. It is assumed that the body of rotation, consisting of that part which has emerged together with its symmetrical reflection in the y-axis, is in a transverse stream. Since this problem is not axi-symmetric, to determine the dipole moments we use the condition that a stream line coincides with a generator of the body.

The potential of dipoles with moments m(x, t) parallel to the y-axis, distributed in the segment $[-\alpha, \alpha]$, is:

$$\varphi = -\frac{1}{4\pi}\frac{\partial}{\partial y}\int\limits_{-\alpha}^{\alpha}\frac{m(x, t)\,dx}{r}, \tag{3}$$

where $r = \sqrt{(x-\xi)^2 + \rho^2}$.

If we superpose the potential of the incident flow $\varphi = V_\infty y$, we obtain the differential equations for the stream lines of the resulting flow [1].

If in the equations for the stream lines we substitute expressions for $\partial\varphi/\partial x$ and $\partial\varphi/\partial\rho$, obtained from (3), we obtain an equation for m(x, t):

$$\frac{1}{4\pi}\left[3y\frac{dy}{dx}\int\limits_{-\alpha}^{\alpha}\frac{m(\xi, t)(x-\xi)\,d\xi}{r^5} + \int\limits_{-\alpha}^{\alpha}\frac{m(\xi, t)\,[(x-\xi)^2 - 2y^2]\,d\xi}{r^5}\right] + V_\infty = 0. \tag{4}$$

If we introduce into (4) the expression for y from the equation for the generators obtained by taking into account the part of the body reflected in the y-axis, we have from (4) the condition for determining the moments of the dipoles for the given flow.

Thus, it follows from what has been said above that the problem of the emergence of a body from a solid wall in translational motion can be reduced to the solution of a Fredholm equation of the first kind. It should be noted that since in this problem the domain of definition of the coordinate x does not coincide with the domain of definition of the variable ξ, additional difficulties occur for this problem which in general is incorrect, but in Tikhonov's papers a sufficiently effective method is proposed for regularizing the Fredholm integral equations of the first kind and this can be used to obtain the solution of the problem to an adequate degree of accuracy. The method must be adapted for each individual case. The method we describe below is simplified, but it makes it possible to solve the problem without using computers.

We consider the solution of the problem of the emergence of a slender body of rotation of arbitrary shape from a wall. We write equations (2) and (4) in elliptic coordinates, linked to the x_1y_1 system by the following equations:

$$x_1 = c\lambda\mu,$$
$$y_1 = c\sqrt{\lambda^2 - 1}\sqrt{1-\mu^2},$$

where $c = kS/k+1$; the choice of the integer k is discussed later.

We have the following integral equation for the source strength:

$$\frac{1}{4\pi}\int\limits_{-1}^{1}\frac{(c\lambda\mu - c\mu')\,q(c\mu')\,dc\mu'}{c\sqrt{\lambda^2+\mu^2-1-2\lambda\mu'\mu+\mu'^2}} = \frac{1}{4\pi}\int\limits_{-1}^{1}q(c\mu')\,dc\mu' + \frac{1}{4}V_0c^2(\lambda^2-1)(1-\mu^2). \tag{5}$$

For a slender body we can assume approximately that $\lambda \approx 1$, from some moment of time onwards. Hence the expression under the radical in (5) can be simplified and we have

$$\frac{1}{4\pi}\int\limits_{-1}^{1}\frac{cq(c\mu')(\lambda\mu-\mu')\,d\mu'}{\mu-\mu'} = \frac{1}{4\pi}\frac{dU}{dt} + \frac{1}{4}V_0c^2(1-\mu^2)(\lambda^2-1). \tag{6}$$

We seek the solution of equation (6) in the form:

$$q(c\mu') = cV_0 \Sigma a_n P_n(\mu'). \tag{7}$$

Substituting the expression for $q(c\mu)$ from (7) into (6), we obtain the following equation for the coefficients:

$$\Sigma a_n \left[\frac{n+1}{2n+1} Q_{n+1}(z) + \frac{n}{2n+1} Q_{n-1}(z)\right] = \frac{dU_T/dt}{2(\lambda-1)c^2V_0} + \frac{\pi}{2}(1-\mu^2)(\lambda+1). \tag{8}$$

where we have used the familiar relation for Legendre functions:

$$\frac{1}{2}\int_{-1}^{1} \frac{P_n(\mu)\,d\mu}{z-\mu} = Q_n(z), \text{where } z = \mu + (\lambda-1)i,$$

By expanding the right side in a series in Legendre functions, after replacing dU/dt and λ with expressions from the equation for the generators and equating the coefficients of like functions Q_n, we obtain the coefficients a_n. If A_n are the coefficients of the expansion of the right side, we have the following equations for the a_n:

$$\begin{aligned} a_0 &= \frac{1}{2}\frac{dU}{dt}\frac{1}{c^2V_0}, \quad a_1 = 3A_0, \\ a_2 &= \frac{5}{2}(A_1 - a_0), \quad a_4 = \frac{9}{4}A_3 - \frac{27}{8}A_1, \\ a_3 &= \frac{7}{3}(A_2 - 2A_0), \quad a_5 = \frac{9A_4 + 16(A_2 - 2A_0)}{15}, \end{aligned} \tag{9}$$

etc.

For the expansion of the functions on the right side of equation (5) in series of Legendre functions, we have to find an ellipse inside the body of rotation outside which all the singular points of the function

$$F(\mu) = \left[\frac{dU_T/dt}{2(\lambda-1)c^2V_0} + \frac{\pi}{2}(1-\mu^2)(\lambda+1)\right]$$

lie.

Outside this ellipse the expansion of the function $F(\mu)$ is valid and the coefficients of the expansion can be found as follows:

$$A_n = \frac{2n+1}{2\pi i}\int_{\Omega} F(z)\,P_n(z)\,dz,$$

where Ω – the contour of integration – is the ellipse outside of which the function F(z) is continuous.

Since Ω is a closed contour, inside which there is a finite number of points of discontinuity of F(z), the A_n are easily determined from the residue theorem. It should be noted that it is frequently convenient to investigate the continuity of F(z) outside a circle of radius $r = (k+1)/k$ and then this function is regular outside an ellipse with foci $k = \pm 1$ and having the equation

$$\frac{\mu^2}{1+\left(\frac{k+1}{k}\right)^2} + \frac{\lambda^2}{\left(\frac{k+1}{k}\right)^2} = 1. \tag{10}$$

From the condition that the body of rotation of variable shape lies outside an ellipse we have to choose the integer k appropriately. It will be shown below, by an actual example, how the expansion is actually made.

We shall discuss the emergence of a slender body of rotation of elliptical shape, the generator of which, in elliptic coordinates, has the equation:

$$\lambda = \frac{e^2(k+1)^2\,\mathrm{Sh} - k^2}{2k(k+1)(\mathrm{Sh}-1)}\mu + \frac{(1-\mathrm{Sh})(k+1)}{2k}\frac{1}{\mu}, \; x > 0, \tag{11}$$

where $\mathrm{Sh} = a/S$. We obtain the following integral equation for q(x):

$$\frac{c}{4\pi}\int_{-1}^{1}\frac{q(c\mu')(\mu\lambda-\mu')\,d\mu'}{\mu-\mu'}=b^2V_0\frac{\mathrm{Sh}-1}{\mathrm{Sh}^2}+\frac{1}{4}V_0c^2(1-\mu^2)(\lambda^2-1). \tag{12}$$

As shown in the previous section, putting

$$q(c\mu')=cV_0\Sigma a_nP_n(\mu'),$$

we can find the coefficients from the following equation:

$$\Sigma a_n\left[\frac{n+1}{2n+1}Q_{n+1}(\mu)+\frac{n}{2n+1}Q_{n-1}(\mu)\right]=\frac{2\pi\left(\frac{1-e^2}{e^2}\right)\frac{\mathrm{Sh}-1}{\mathrm{Sh}^2}}{\lambda}+\frac{\pi}{2}(1-\mu^2)(\lambda^2-1). \tag{13}$$

We replace λ by μ on the right side of (9), and expand the first part, denoted by $f(\mu)$, in Legendre functions $Q_n(\mu)$. Let

$$f(\mu)=2\pi\left(\frac{1-e^2}{e^2}\right)\frac{\mathrm{Sh}-1}{\mathrm{Sh}^2}\frac{1}{A\mu+\frac{B}{\mu}-1}+\frac{\pi}{2}(1-\mu^2)\left(A\mu+\frac{B}{\mu}+1\right). \tag{14}$$

Then we have the following equation for the coefficients of the expansion $f(\mu)=\Sigma A_nQ_n$:

$$A_n=\frac{2n+1}{2\pi i}\int_{\Omega}f(z)P_n(z)\,dz,$$

where

$$\Omega:\frac{\mu^2}{1+\left(\frac{k+1}{k}\right)^2}+\frac{\lambda^2}{\left(\frac{k+1}{k}\right)^2}=1;\ z=\mu+i(\lambda-1);$$

$$\left.\begin{aligned}A&=\frac{e^2(k+1)^2\mathrm{Sh}^2-k^2}{2k(k+1)(\mathrm{Sh}-1)}\\ B&=\frac{(1-\mathrm{Sh})(k+1)}{2k}\end{aligned}\right\}x>0;\qquad \left.\begin{aligned}A&=\frac{e^2(k+1)^2\mathrm{Sh}^2-k^2}{2k(k+1)(\mathrm{Sh}-3)}\\ B&=\frac{(3-\mathrm{Sh})(k+1)}{2k}\end{aligned}\right\}x<0. \tag{15}$$

Putting $P=[(1-e^2)/e^2](\mathrm{Sh}-1)/\mathrm{Sh}^2$, we obtain an expression for the coefficients:

$$A_0=\frac{1}{i}\left[P\int_{\Omega}\frac{zdz}{Az^2+B-z}+\frac{1}{4}\int_{\Omega}\frac{(1-z^2)(Az^2+B-z)\,dz}{z}\right],$$

$$A_1=\frac{3}{i}P\int_{\Omega}\frac{z^2dz}{Az^2+B-z},$$

$$\cdots\cdots\cdots\cdots$$

$$A_n=\frac{2n+1}{i}\left[P\int_{\Omega}\frac{P_n(z)\,zdz}{Az^2+B-z}+\frac{1}{4}\int_{\Omega}\frac{P_n(z)(1-z^2)(Az^2+B-z)}{z}dz\right].$$

Using the method of residues for the integrals in the expressions for the coefficients A_n, and taking into account the singular points of the function $f(z)$, we have the following equations:

$$A_0=2\pi\left[P\Sigma\frac{z_i(z-z_i)}{Az^2+B-z/z_i}+\frac{1}{4}B\right],$$

$$A_1=6\pi P\Sigma\frac{z_i^2(z-z_i)}{Az^2+B-z/z_i}, \tag{16}$$

etc.

If equation (11) defines the contour Ω, we have to choose the integer k from the following condition: $(2k+1)/2k^2\geq(1/e^2-1)/2$, which is obtained from the condition for determining the generator of the body of rotation outside the ellipse (10).

Using (6) we can write an expression for the coefficients in the expansion of the function $q(x_1)=\Sigma a_iP_i\cdot(x_1/c)$.

By finding the distribution of the source intensities and locating the reflected sources symmetrically with respect to the y-axis, we obtain the following expression for the potential φ:

$$\varphi(x_1 y_1 z_1) = -\frac{cV_0}{4\pi}\left[\int_{-1}^{1}\frac{\Sigma a_i P_i(\mu)\,d\mu}{\sqrt{\left(\mu+\frac{S}{c}+\frac{x_1}{c}\right)^2+\frac{\rho_1^2}{c^2}}}+\int_{-1}^{1}\frac{\Sigma a_i P_i(\mu)\,d\mu}{\sqrt{\left(\mu-\frac{x_1}{c}\right)^2+\frac{\rho^2}{c^2}}}\right]. \tag{17}$$

If we denote the quantity in brackets by I_i, we obtain

$$\varphi(x_1 y_1 z_1) = -\frac{cV_0}{4\pi}\Sigma a_i I_i.$$

If we consider the emergence of a body of rotation without taking into account the motion of the wall, we can put one of the coordinates equal to zero (z = 0) and so (17) is simplified and the pressure distribution on the surface of the body can be found from the equation

$$P - P_\infty = -\left[\rho_0\frac{\partial\varphi}{\partial t}+\frac{\rho_0}{2}(\operatorname{grad}\varphi)^2\right]. \tag{18}$$

As indicated above, the potential of the flow when the wall is in motion can be calculated independently of the calculation of the emergence of the body and, for example, Serebriiskii's method [3] can be used to compute the transverse flow round a slender body of rotation. We shall not discuss this method in detail.

We turn now to the determination of the forces on the vertically emerging body of rotation. For this we use the equation for the momentum obtained by Sedov [2]:

$$\bar{Q} = \rho_0\iint_S \varphi\bar{n}\,dS = -\rho_0 UV^* - \rho_0\iiint_U \bar{r}\operatorname{div}V\,d\tau + 4\pi\rho_0\bar{e},$$

where

$$\bar{e} = e_1\bar{i}+e_2\bar{j};\ e_1 = -\frac{1}{4\pi}\int_{-\alpha}^{\alpha} q(\mu)\,d\mu;\ e_2 = -\frac{1}{4\pi}\int_{-\alpha}^{\alpha} m(\mu)\,d\mu.$$

Since in the case when the body emerges perpendicularly to the wall we have

$$\iiint_U \bar{r}\operatorname{div}V\,d\tau = \bar{i}\int_{-\alpha}^{\alpha} xq(x)\,dx,$$

it follows that

$$\bar{Q} = -\rho_0 UV^* + 4\pi\rho_0\bar{e} + i\rho_0 e_1 4\pi.$$

Since when the body emerges from the solid wall there are two distributions of sources, situated symmetrically with respect to the y-axis, we obtain

$$e_1 = -\frac{Vc}{4\pi}\int_{-\alpha}^{\alpha} xq(x)\,dx = \frac{Vc^2S}{4\pi}\int_{-1}^{1}\Sigma a_i P_i(\mu)\,d\mu.$$

Since a_1 is not a function of μ, we have

$$e_1 = \frac{Vc^2S}{4\pi}\Sigma a_i I_i^0,$$

where

$$I_i^0 = \int_{-1}^{1} P_i(\mu)\,d\mu.$$

Knowing the equation for the momentum, we obtain the total force acting on the system of bodies inside some closed volume U_0 in the form:

$$\bar{X} = \frac{d\bar{Q}}{dt} = -\rho_0\left(\frac{dU}{dt}\bar{V}^* - \frac{d\bar{V}^*}{dt} - U\right) + i\,8\pi\rho_0\frac{de_1}{di}.$$

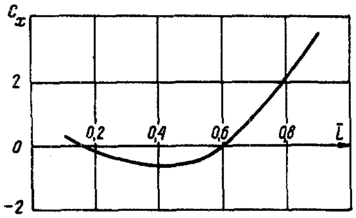

Fig. 2. The computed curve for the drag coefficient C_X as a function of the length of the part of the body of rotation which has emerged divided by the total length of the body. C_X is divided by $\rho_0 V^2 \pi a b/2$.

If we separate from this total the force acting on the boundary, we obtain the force on the body. We calculate the forces on the boundary by integrating the pressure on the surface of the boundary, obtained from Lagrange's equation:

$$\bar{X}_b = -\rho_0 \iint\limits_{S_b} \frac{\partial \varphi}{\partial t} \bar{n} dS - \frac{\rho_0}{2} \iint\limits_{S_b} \left(\frac{\partial \varphi}{\partial \rho} \right)^2 \bar{n} dS,$$

where S_b is the surface of the boundary. We have taken into account the fact that $\partial\varphi/\partial x = 0$ on the surface of the boundary. In the first approximation we have

$$X_b = \frac{\rho_0}{4\pi} \left[\iint \left(\frac{\partial Vc}{\partial t} \sum_{i=0}^{1} a_i I_i + \frac{\partial \sum_{i=0}^{1} a_i I_i}{dt} cV \right) dS + \frac{cV}{2} \iint \left(\frac{\partial}{\partial \rho} \sum_{i=0}^{1} a_i I_i \right)^2 dS \right]. \tag{19}$$

Calculations for the case $a/b = 10$ and $V = Wt$, using the above equations, gave the results shown in Fig. 2, where the function $C_X(Sh)$ is shown, the forces being divided by $\rho_0 V^2 \pi a b/2$.

As was indeed to be expected, in the limiting cases (at the beginning of emergence and at its end) the greatest errors occur. The positive forces arising as the body emerges are evidently to be explained by the effect of the wall.

NOTATION

X, Y	are the fixed coordinate system attached to the solid wall;
X_1, Y_1	are the coordinate system attached to the body (moving relative to the XY system with velocity $1/2V_0$), $x_1 = x - S$, $S = 1/2 \int V_0 dt$, $y_1 = y$;
λ, μ	are the elliptic coordinates: $\lambda = \mathrm{ch}\,\xi$, $x_1 = c\,\mathrm{ch}\,\xi \cos\eta$; $\mu = \cos\eta$, $y_1 = c\,\mathrm{sh}\,\xi \sin\eta$;
$q(x)$	is the strength of distributed sources;
$m(x)$	is the dipole strength;
V_0	is the rate of emergence of body from wall;
W	is the acceleration of body;
V_∞	is the wall velocity;
Sh	is the Strouhal number;
t	is the time;
P_n	is the Legendre polynomial;
Q_n	is the Legendre function;
c	is the coordinate of the focus of the ellipse;
k	is the integer greater than unity;
e	is the eccentricity of the ellipse;
b	is the minor semi-axis of the ellipsoid of rotation;
a	is the major semi-axis of the ellipsoid of rotation;
$\bar{Q}$	is the momentum;
V^*	is the velocity of the center of gravity of the body;
U	is the volume of the part of the body of rotation which has emerged;
$\bar{X}$	is the force along the x-axis;
X_b	is the force on the boundary;
X_B	is the force on the body;
X	is the total force on the system;
ρ_0	is the density of the liquid.

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